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Elementary Darboux transformations for the n-component KP-hierarchy

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Abstract

In this paper a purely algebraic setting is described in which a characterization of the dual wavefunctions of the multicomponent KP-hierarchy and an interpretation of the bilinear form of this system of nonlinear equations can be given. The framework enables the construction of solutions starting from a matrix version of the Sato Grassmannian and the expression in formal power series determinants, the so-called τ-functions. This leads to a geometric description of the elementary Darboux transformations for the n-component KP-hierarchy and one concludes with showing how to construct them, both at the differential operator level as at the τ-function level.

Subject classification: 22E65, 22E70, 35Q58, 58B25.

Keywords: n-component KP hierarchy, dual wavefunction, bilinear relations, Darboux transformations.

1 Introduction

Recall that the KP-hierarchy is a tower of non linear differential equations in infinitely many variables $\{x, t_1, \ldots, t_n, \ldots\}$. A convenient formulation of these equations is as a set of Lax equations for a scalar pseudodifferential operator L. The coefficients of L belong to a ring R of functions in the parameters $\{x, t_n\}$ that is stable w.r.t. the privileged derivation $\partial = \dfrac{\partial}{\partial x}$ and all the derivations $\{\partial_n := \dfrac{\partial}{\partial t_n}, n \geq 1\}$. The operator L has the form

$$L = \partial + \sum_{j<0} l_j \partial^j \quad , \qquad \text{with } \ l_j \in R. \tag{1}$$

The equations of the KP-hierarchy are then

$$\partial_n(L) = [(L^n)_+, L], \tag{2}$$

where $(L^n)_+$ denotes the differential operator part of the pseudodifferential operator L^n. Note that equation (2) for $n = 1$ reads $\partial_1(L) = \partial(L)$. Therefore it is customary to take $x = t_1$. The n-component KP-hierarchy or multicomponent KP-hierarchy is a matrix version of the KP-hierarchy that made its first appearance in the work of the Sato school, see [DJKM] and [UT]. In this matrix version the variables are the $\{x, t_{i\alpha} | i \geq 1, 1 \leq \alpha \leq n\}$. We write again ∂ for the differentiation w.r.t. the parameter x and likewise $\partial_{i\alpha}$ for the differentiation w.r.t. $t_{i\alpha}$. The pseudodifferential operators in ∂ occurring in the equations have coefficients belonging to the ring $M_n(R)$, where R is some ring of functions in the parameters $\{x, t_{i\alpha} | i \geq 1, 1 \leq \alpha \leq n\}$ that is stable w.r.t. ∂ and the $\{\partial_{i\alpha}\}$. We will use the notation $M_n(R)[\partial, \partial^{-1})$ for this ring. Instead of just one operator L, we have $n+1$ pseudodifferential operators $\{L, U_\alpha | 1 \leq \alpha \leq n\}$. They have the form

$$L = \partial + \sum_{j \geq 1} l_j \partial^{-j} \text{ and } U_\alpha = E_{\alpha\alpha} + \sum_{j \geq 1} u_{\alpha,j} \partial^{-j}. \tag{3}$$

Here all the l_j and $u_{\alpha,j}$ belong to $M_n(R)$ and $E_{\alpha\beta}$ is the $n \times n$-matrix with its (α, β)-th entry equal to 1 and the other entries equal to zero. This set of operators should satisfy the following system of nonlinear equations inside $M_n(R)[\partial, \partial^{-1})$

$$[L, U_\alpha] = [U_\alpha, U_\beta] = 0, \tag{4}$$

$$\partial_{i\alpha}(L) = [(L^i U_\alpha)_+, L], \tag{5}$$

$$\partial_{i\alpha}(U_\beta) = [(L^i U_\alpha)_+, U_\beta], \tag{6}$$

for all α and β in $\{1, \ldots, n\}$. If L and the U_α are solutions of these equations and $P \in M_n(R)[\partial, \partial^{-1})$ has e.g. the form $P = \mathrm{Id} + \sum_{j<0} p_j \partial^j$, then the operators

$$L_P := PLP^{-1} \text{ and } (U_\alpha)_P := PU_\alpha P^{-1} \tag{7}$$

in $M_n(R)[\partial, \partial^{-1})$ have the same form again as L and the U_α, but they do not have to satisfy the equations (4), (5) and (6). For any invertible operator $P \in M_n(R)[\partial, \partial^{-1})$ such that conjugation with P transforms solutions of the equations (4), (5) and (6) into solutions , we call the transformation (7) a *Darboux transformation* of the n-component KP-hierarchy. They are used to construct new solutions out of a given one. Recently, see ([LM]), it was found that matrix hierarchies such as the multicomponent KP-hierarchy are an important mean to construct solutions of a non linear system from two-dimensional topological field theory, the so-called $WDVV$-equations. A future intent is to use the construction presented here also in that direction.

In the present paper the so-called *elementary Darboux transformations* be described, which correspond to a first order differential operator $P \in M_n(R)$ and which are the building blocks for more general operators, see [HL01] for the case of the KP-hierarchy. The content of this paper is as follows: we start with a description of the algebraic background of this system of equations and show how they result from their linearization. Next the form of the eigenfunctions occurring in this linear system is treated. They determine the solutions of the non linear equations completely and are called wavefunctions. Associated with them is an eigenfuction of another linear system, the so-called dual wave function, which forms the subject of the next section. In particular a useful characterization of these dual wave functions is given and the duality between both systems is ed. A wide class of wave functions can be constructed from planes in the Grassmann manifold of Sato adapted to this vector situation. This is the subject of the subsequent section. Next these wave functions and their duals are expressed in the so-called τ-functions. The final section is devoted to the geometric description and the construction of the elementary Darboux transformations. Here one also finds the description of the transformation formulae on the level of the τ-functions.

2 The algebraic set-up

The compact form in which the equations of the n-component KP-hierarchy are presented in the introduction, see also [UT], is the so-called Lax form. To give some insight in this form, we describe here shortly the underlying algebraic structure in the style of [W]. Note that the equations (4), (5) and (6) boil down to an infinite number of polynomial relations between the matrixcoefficients $\{(l_j)_{\rho\sigma}\}$ and $\{(u_{\alpha,j})_{\rho\sigma}\}$ of the l_j and $u_{\alpha,j}$ and their derivatives w.r.t. ∂ and the $\partial_{i\alpha}$. Therefore we start with the situation that no relations are present and we consider the following two collections of commuting unknowns $\{(u^{(i)}_{\alpha,j})_{\rho\sigma}\}$ and $\{(l^{(i)}_j)_{\rho\sigma}\}$, where $j \geq 1,\ i \geq 0$ and the indices $\alpha,\ \rho$ and $\sigma \in \{1, \dots, n\}$. Now let A be the ring generated by them, i.e.

$$A = \mathbb{C}[(u^{(i)}_{\alpha,j})_{\rho\sigma}, (l^{(i)}_j)_{\rho\sigma}].$$

Since there are no relations inside A, there is a unique $\mathbb{C}$-linear derivation $\tilde{\partial} : A \to A$ such that

$$\tilde{\partial}((u^{(i)}_{\alpha,j})_{\rho\sigma}) = (u^{(i+1)}_{\alpha,j})_{\rho\sigma} \text{ and } \tilde{\partial}((l^{(i)}_j)_{\rho\sigma}) = (l^{(i+1)}_j)_{\rho\sigma}$$

for all $i \geq 0$, $j \geq 1$ and all α, ρ and σ in $\{1, \dots, n\}$. Any derivation Δ of a commutative $\mathbb{C}$-algebra B extends in a natural way to a derivation Δ of the $\mathbb{C}$-algebra $M_n(B)$ by putting $\Delta(b)_{ij} = \Delta(b_{ij})$ for all $b = (b_{ij}) \in M_n(B)$. In the ring $M_n(A)[\tilde{\partial}]$ of differential operators in $\tilde{\partial}$ with coefficients from $M_n(A)$, one can, in general, not take roots of monic operators. Thereto one passes to the extension $M_n(A)[\tilde{\partial}, \tilde{\partial}^{-1})$ of all pseudodifferential operators in $\tilde{\partial}$ with coefficients from $M_n(A)$. It consists of all expressions

$$\sum_{i=-\infty}^{N} a_i \tilde{\partial}^i, a_i \in M_n(A) \text{ for all } i,$$

that are added coefficientwise and that are multiplied according to

$$\tilde{\partial}^j \cdot a\tilde{\partial}^i = \sum_{k=0}^{\infty} \begin{pmatrix} j \\ k \end{pmatrix} \tilde{\partial}^k(a)\tilde{\partial}^{i+j-k}$$

Each operator $A = \sum a_j \tilde{\partial}^j$ decomposes as $A = A_+ + A_-$ with $A_+ = \sum_{j\geq 0} a_j \tilde{\partial}^j$ its differential operator part and $A_- = \sum_{j<0} a_j \tilde{\partial}^j$ its pure integral operator part. Note that in order to define a derivation of A that commutes with $\tilde{\partial}$, it suffices to prescribe the image of all the $\{(u^{(0)}_{\alpha,j})_{\rho\sigma}\}$ and all the $\{(l^{(i)}_j)_{\rho\sigma}\}$ and this can be done freely. The same thing holds for derivations of $M_n(A)$ that should commute with $\tilde{\partial}$ in $\mathrm{Der}(M_n(A))$. The specific choices that we are interested in here, can in a condense way be expressed in the operators $\tilde{L}$ and $\tilde{U_\alpha}$ from $M_n(A)[\tilde{\partial}, \tilde{\partial}^{-1})$ defined by

$$\tilde{L} = \tilde{\partial} + \sum_{j\geq 1} l^{(0)}_j \tilde{\partial}^{-j} \text{ and } \tilde{U_\alpha} = E_{\alpha\alpha} + \sum_{j\geq 1} u^{(0)}_{\alpha,j} \tilde{\partial}^{-j}.,$$

Here $l_j^{(0)}$ resp. $u_{\alpha,j}^{(0)}$ are the elements of $M_n(A)$ with the (ρ,σ)-entry equal to $(l_j^{(0)})_{\rho\sigma}$ resp. $(u_{\alpha,j}^{(0)})_{\rho\sigma}$. For each $i \geq 1$ and each α, $1 \geq \alpha \geq n$, let $\tilde{\partial}_{i\alpha}$ be the unique derivation of A resp. $M_n(A)$ that commutes with $\tilde{\partial}$ and that is fixed by the following operator identities in $M_n(A)[\tilde{\partial},\tilde{\partial}^{-1})$

$$\tilde{\partial}_{i\alpha}(\tilde{L}) := 0\tilde{\partial} + \sum_{j>0} \tilde{\partial}_{i\alpha}(l_j^{(0)})\tilde{\partial}^{-j} = [((\tilde{L})^i\tilde{U}_\alpha)_+, \tilde{L}], \tag{8}$$

$$\tilde{\partial}_{i\alpha}(\tilde{U}_\beta) := \tilde{\partial}_{i\alpha}(E_{\beta\beta}) + \sum_{j>0} \tilde{\partial}_{i\alpha}(u_{\beta,j}^{(0)})\tilde{\partial}^{-j} = [((\tilde{L})^i\tilde{U}_\alpha)_+, \tilde{U}_\beta]. \tag{9}$$

Since we think of the derivations $\tilde{\partial}_{i\alpha}$ as potential derivatives w.r.t. independent variables $\{t_{i\alpha}\}$, we like them to commute. As in [W] it can be shown that these derivations commute among each other, if we have for all α and β

$$[\tilde{L}, \tilde{U}_\alpha] = [\tilde{U}_\alpha, \tilde{U}_\beta] = 0 \tag{10}$$

The complex of equations (8), (9) and (10) is called the n-component KP-hierarchy. One is mainly interested in the equations (8) and (9), because they render the non linear differential equations, and they are called the Lax equations of the hierarchy. To give sense to this exercition, the next step has to be to make concrete realizations of the relations in (8), (9) and (10). This means that we are looking first of all for commutative $\mathbb{C}$-algebras R equiped with a privileged derivation $\partial : R \to R$ and a collection of derivations $\{\partial_{i\alpha}, i \geq 1\, 1 \leq \alpha \leq n\}$ of R that commute all with ∂ and also among each other. Further there should be a $\mathbb{C}$-algebra morphism $\pi : A \to R$ that is compatible with all these derivations i.e. it should satisfy

$$\pi \circ \tilde{\partial} = \partial \circ \pi \ \text{ and } \ \pi \circ \tilde{\partial}_{i\alpha} = \partial_{i\alpha} \circ \pi, \ \text{ with } \ i \geq 1, 1 \leq \alpha \leq n. \tag{11}$$

As mentioned in the introduction, one should think of R as some ring of functions depending of the parameters $\{x, t_{i\alpha} | i \geq 1, 1 \leq \alpha \leq n\}$, where the derivation ∂ is differentiation w.r.t. the parameter x and likewise $\partial_{i\alpha}$ is differentiation w.r.t. $t_{i\alpha}$. Since all the unknowns $\{(l_j^{(i)})_{\rho\sigma}\}$ and $\{(u_{\alpha,j}^{(i)})_{\rho\sigma}\}$ are independent, this allows you to pick their images freely to get a $\mathbb{C}$-algebra morphism $\pi : A \to R$.

To get the first property in (11) for such a morphism π, one still has the freedom of choosing the image of the $\{(l_j^{(0)})_{\rho\sigma}\}$ and the $\{(u_{\alpha,j}^{(0)})_{\rho\sigma}\}$'s freely. In order that the remaining properties hold, the morphism π has to factorize over the relations in (8), (9) and (10). In other words, the operators L and U_α, $1 \leq \alpha \leq n$, in $M_n(R)[\partial,\partial^{-1})$ given by

$$L = \partial + \sum_{j\geq 1} \pi(l_j^{(0)})\partial^{-j} \ \text{ and } \ U_\alpha = E_{\alpha\alpha} + \sum_{j\geq 1} \pi(u_{\alpha,j}^{(0)})\partial^{-j}. \tag{12}$$

should satisfy the following system of nonlinear equations inside $M_n(R)[\partial,\partial^{-1})$

$$[L, U_\alpha] = [U_\alpha, U_\beta] = 0, \tag{13}$$

$$\partial_{i\alpha}(L) = [(L^iU_\alpha)_+, L], \tag{14}$$

$$\partial_{i\alpha}(U_\beta) = [(L^iU_\alpha)_+, U_\beta], \tag{15}$$

for all α and β in $\{1,\ldots,n\}$ and these are precisely the equations from the introduction. Given R, δ and the $\delta_{i\alpha}$, we call a set of operators L and U_α, $1 \leq \alpha \leq n$, in $M_n(R)[\partial,\partial^{-1})$ of the form (12) a *solution* of the n-component KP-hierarchy. One can see a solution L and $\{U_\alpha\}$ of the n-component KP-hierarchy as a deformation of the trivial solution $L = \partial$ and $U_\alpha = E_{\alpha\alpha}, 1 \leq \alpha \leq n$.

As in the one-dimensional case, there exists a linearization of the n-component KP-hierarchy from which the Lax equations (14) and (15) follow as compatibility conditions. Consider namely

$$L\psi = z\psi \quad , \quad U_\alpha \psi = \psi E_{\alpha\alpha} \tag{16}$$

and

$$\partial_{i\alpha}(\psi) = (L^i U_\alpha)_+(\psi) =: P_{i\alpha}(\psi) \qquad \text{for all } \ i \geq 1, 1 \leq \alpha \leq n. \tag{17}$$

If one applies the operator $\partial_{i\alpha}$ to both sides of both equations in (16) and performs the following manipulations

$$\partial_{i\alpha}(L)\psi + L\partial_{i\alpha}(\psi) = \{\partial_{i\alpha}(L) + LP_{i\alpha}\}\psi = z\partial_{i\alpha}(\psi) = P_{i\alpha}(z\psi) = \{P_{i\alpha}L\}(\psi), \tag{18}$$

$$\partial_{i\alpha}(U_\beta)\psi + U_\beta \partial_{i\alpha}(\psi) = \{\partial_{i\alpha}(U_\beta) + U_\beta P_{i\alpha}\}\psi = \partial_{i\alpha}(\psi)E_{\beta\beta} = \{P_{i\alpha}U_\beta\}(\psi), \tag{19}$$

then one ends up with the equations

$$(\partial_{i\alpha}(L) - [(L^i U_\alpha)_+, L])(\psi) = 0 \text{ and } (\partial_{i\alpha}(U_\beta) - [(L^i U_\alpha)_+, U_\beta])(\psi) = 0 \tag{20}$$

So, if we can scratch the function ψ in both expressions, we get the equations (14) and (15) for L and the $\{U_\alpha\}$. Once one has obtained with this procedure operators L and U_α that satisfy (14) and (15), one still has to verify the equations (13). This will be immediate from our construction. We proceed now with specifying somewhat more the functions ψ in the linearization. This is done in the next section.

3 Oscillating matrix functions

From the equations (16) and (17), we see that the functions ψ must be susceptible for left actions of various pseudodifferential operators with coefficients from $M_n(R)$ and for right actions with elements from $M_n(\mathbb{C})$. Therefore it is not surprising that they will be matrix functions. To motivate the form of the matrix functions ψ that we will choose, we consider the trivial solution $L = \partial$ and $U_\alpha = E_{\alpha\alpha}$. Then the equations (16) reduce to

$$\partial(\psi) = z\psi \quad , \quad E_{\alpha\alpha}\psi = \psi E_{\alpha\alpha} \quad \text{and} \quad \partial_{i\alpha}(\psi) = z^i E_{\alpha\alpha}\psi. \tag{21}$$

Hence, for the trivial solution of the n-component KP-hierarchy, the matrix function

$$\gamma(x, t, z) = \exp\left(xz + \sum_{\alpha=1}^{n}\sum_{i=1}^{\infty} t_{i\alpha}E_{\alpha\alpha}z^i\right) = \sum_{k=0}^{\infty} \underline{p}_k(x, t)z^k, \tag{22}$$

satisfies the equation in (16). Note that in the definition of $\gamma(x, t, z)$ we used the short-hand notation $xz := xI_n z$. Next we introduce a space of matrix functions for which all the manipulations at the end of section 2 make sense and whose elements can be seen as perturbations of the solution (22). This space M of so-called oscillating matrix functions, is defined as

$$M = \left\{ \left(\sum_{j\leq N} a_j z^j\right) \exp\left(xz + \sum_{\alpha=1}^{n}\sum_{i=1}^{\infty} t_{i\alpha}E_{\alpha\alpha}z^i\right) \middle| a_j \in M_n(R) \right\}.$$

Note that we will treat in this paper the product in the elements of M of a factor meromorphic at infinity and the exponential factor corresponding to the trivial solution as a formal one. It would no longer be a formal matter, if one can make sense in $M_n(R)$ of all the

$$\sum_{k\in\mathbb{Z}} a_{l-k}\underline{p}_k, l \in \mathbb{Z}, \tag{23}$$

and this clearly requires convergence considerations. They have been given in ([HP]). Here we present a more general algebraic approach.

The space M can be made into a module for the ring $M_n(R)[\partial, \partial^{-1})$ of pseudodifferential operators in ∂ with coefficients from $M_n(R)$. If one puts the actions of the elements $b \in M_n(R)$ and the derivations ∂ and $\partial_{k\beta}$ as follows

$$b\left[\left(\sum_j a_j z^j\right)\exp\left(xz + \sum_{\alpha=1}^n \sum_{i\geq 1} t_{i\alpha}E_{\alpha\alpha}z^i\right)\right] = \left(\sum_j ba_j z^j\right)\gamma(x,t,z)$$

$$\partial\left\{\left(\sum_j a_j z^j\right)\exp\left(xz + \sum_{\alpha=1}^n \sum_{i\geq 1} t_{i\alpha}E_{\alpha\alpha}z^i\right)\right\} = \left(\sum_j \partial(a_j) + \sum_j a_j z^{1+j}\right)\gamma(x,t,z),$$

$$\partial_{k\beta}\left\{\left(\sum_j a_j z^j\right)\exp\left(xz + \sum_{\alpha=1}^n \sum_{i\geq 1} t_{i\alpha}E_{\alpha\alpha}z^i\right)\right\} = \left(\sum_j \partial_{k\beta}(a_j) + \sum_j a_j E_{\beta\beta}z^{k+j}\right)\gamma(x,t,z),$$

then one sees immediately that ∂ acts by an invertible transformation on M. Hence these actions extend in a natural way to one of $M_n(R)[\partial, \partial^{-1})$. Thus M becomes even a free left $M_n(R)[\partial, \partial^{-1})$-module, for

$$\left(\sum p_j\partial^j\right)\cdot\exp\left(xz + \sum_{\alpha,i} t_{i\alpha}E_{\alpha\alpha}z^i\right) = \left(\sum p_j z^j\right)\exp\left(xz + \sum_{\alpha,i} t_{i\alpha}E_{\alpha\alpha}z^i\right).$$

M is also in a natural way a right module for the diagonal matrices in $M_n(\mathbb{C})$. For, if $A = \sum_{i=1}^n a_i E_{ii}, a_i \in \mathbb{C}$, then we put

$$\left\{\sum a_j z^j\right\}\exp\left(xz + \sum t_{i\alpha}z^iE_{\alpha\alpha}\right)\cdot A := \left\{\sum_j a_j A z^j\right\}\exp\left(xz + \sum t_{i\alpha}z^iE_{\alpha\alpha}\right).$$

Since all the operations in the formulae (16) and (17) now make sense in M, one can look for solutions of these equations inside M. Note that the equations (16) impose still another relation on the operators U_α. For, we have

$$(\sum_{\alpha=1}^n U_\alpha)\psi = \psi(\sum_{\alpha=1}^n E_{\alpha\alpha}) = \psi, \tag{24}$$

and, since M is a free $M_n(R)[\partial, \partial^{-1})$-module, this implies that

$$\sum_{\alpha=1}^{n} U_\alpha = I_n. \tag{25}$$

Hence the solutions of the n-component KP-hierarchy that one obtains via the linearization satisfy still this additional condition.

There are two types of transformations that map solutions ψ in M of the equations (16) and (17) into new ones for the same operators L and U_α. First there is the right action of the diagonal matrices in $M_n(\mathbb{C})$. Secondly there is an action of a discrete group. For, if $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{Z}^n$ and we write z^δ for the element

$$\operatorname{diag}\left(z^{\delta_1}, \dots, z^{\delta_n}\right) = \begin{pmatrix} z^{\delta_1} & & 0 \\ & \ddots & \\ 0 & & z^{\delta_n} \end{pmatrix}.$$

Note that the element ψz^δ belongs to M for all $\psi \in M$ and if $\psi = \gamma(x,t,z)P \cdot \exp(xz + \sum t_{i\alpha} z^i E_{\alpha\alpha})$ satisfies the equations (16) and (17) for some L and U_α, then

$$\psi z^\delta = P \cdot z^\delta \cdot \exp(xz + \sum t_{i\alpha} z^i E_{\alpha\alpha}). \tag{26}$$

satisfies the same equations with the same L and U_α.

Any matrix function ψ in M can without any restriction be written in the form (26) with P of the form $P = \sum_{j \le 0} p_j \partial^j$. We assume now that the leading coefficient of P belongs to $GL_n(R)$. The operator P is then invertible and we look in that way at the group orbit of the trivial solution $z^\delta \gamma(x,t,z)$. From the equations (16) for ψ one shows

$$L = P \partial P^{-1},\ U_\alpha = P E_{\alpha\alpha} P^{-1}. \tag{27}$$

Hence we have that L and the $\{U_\alpha\}$ are completely determined by ψ and from these expressions it is also clear that the operators L and the U_α all commute. The equations (16) and (17) for ψ also imply that the leading coefficient of P is a constant for all the derivations $\partial_{i\alpha}$ and ∂ and moreover has to be a diagonal matrix. One can gauge the leading coefficient of the part of ψ that is meromorphic at infinity in various ways. We choose the gauge from [KvdL] here. It requires some notations. Let $\epsilon_1 = (1,0,0\dots,0),\ \epsilon_2 = (0,1,0,\dots,0),\ \dots,\ \epsilon_n = (0,0,\dots,0,1)$ be the standard basis of $\mathbb{Z}^n$. We define a symmetric bilinear form on this lattice by taking $(\epsilon_i|\epsilon_j) = \delta_{ij}$. Let $\varepsilon :\ \mathbb{Z}^n \times \mathbb{Z}^n \to \{\pm 1\}$ be the bimultiplicative function which is defined as

$$\varepsilon(\epsilon_i, \epsilon_j) = \varepsilon_{ij}, \tag{28}$$

with

$$\varepsilon_{ij} = \begin{cases} -1 & \text{if } i > j \\ 1 & \text{if } i \le j. \end{cases} \tag{29}$$

For each vector δ in the lattice we introduce the matrix

$$R(\delta; z) = R^+(\delta; z) := \sum_{i=1}^{n} \varepsilon(\varepsilon_i, \delta) E_{ii} z^{(\varepsilon_i|\delta)} = diag(\varepsilon(\varepsilon_1, \delta), \dots, \varepsilon(\varepsilon_n, \delta)) z^\delta. \tag{30}$$

In the lattice point $\delta \in \mathbb{Z}^n$ we look for oscillating matrix functions ψ^+ of the form

$$\psi^+(\delta; x, t, z) = P^+(\delta; x, t, \partial) \cdot R^+(\delta; z)\gamma(x, t, z) \tag{31}$$

$$=: P^+(\delta; x, t, \partial) \cdot R^+(\delta; \partial) \cdot \gamma(x, t, z) \tag{32}$$

$$\text{with } P^+(\delta; x, t, \partial) = I_n + \sum_{j<0} p_j \partial^j. \tag{33}$$

This gauge carries a special name: such an element ψ^+ in M is called an oscillating matrix function *of type* δ. The connection between oscillating matrix functions of type δ satisfying (16) and (17) and solutions of the n-component KP-hierarchy is given by

Proposition 3.1. *Let* $\psi = P(\delta; x, t, \partial) \cdot R^+(\delta; \partial) \cdot \gamma(x, t, z) = P.R^+(\delta; \partial) \cdot \gamma(x, t, z)$ *in* M *be an oscillating matrix function of type* δ *. Assume, we have operators* $L = L(\delta; x, t, \partial)$ *and* $U_\alpha = U_\alpha(\delta; x, t, \partial), 1 \leq \alpha \leq n$*, in* $M_n(R)[\partial, \partial^{-1})$ *of the form (12), such that the equations in (16) hold and such that for* $i \geq 1$ *and all* $\beta, 1 \leq \beta \leq n$*, there holds*

$$\partial_{i\beta}(\psi) = B_{i\beta}(\psi) \text{ for some } B_{i\beta} \in M_n(R)[\partial].$$

Then we have $L = P\partial P^{-1}$*,* $U_\alpha = PE_{\alpha\alpha}P^{-1}$ *and* $B_{i\beta} = P_{i\beta} = (L^i U_\beta)_+$ *for all* i *and* β*. Moreover the operators* L *and the* $\{U_\alpha\}$ *are a solution of the* n*-component* KP*-hierarchy.*

For a proof we refer the reader to [HL00]. A useful consequence of this proposition is

Corollary 3.1. *An oscillating matrix function* $\psi = PR(\delta; z) \exp(xz + \sum t_{i\alpha} z^i E_{\alpha\alpha})$ *of type* δ*, satisfies equation (17) if and only if* P *satisfies the so-called Sato-Wilson equations:*

$$\partial_{i\alpha}(P)P^{-1} = -(P\partial^i E_{\alpha\alpha} P^{-1})_-, \text{ for all } i \geq 1, 1 \leq \alpha \leq n. \tag{34}$$

Such a function ψ *is called a wavefunction of the* n*-component* KP*-hierarchy.*

4 The dual wave function

At the description of the bilinear form of the n-component KP-hierarchy and at the construction of Darboux transformations of this system of equations it is convenient to have besides M also at one's disposal its adjoint space M^* consisting of all formal products

$$\{\sum_{j \leq N} a_j z^j\} \exp(-xz - \sum_{\alpha=1}^{n} \sum_{i=1}^{\infty} t_{i\alpha} z^i E_{\alpha\alpha}) = \{\sum_{j \leq N} a_j z^j\} \gamma(-x, -t),$$

where $a_j \in M_n(R)$ for all j. On the space M^* we define the following actions of $M_n(R)$, ∂ and the $\{\partial_{k\beta}\}$

$$\begin{array}{rcl} b\{(\sum_j a_j z^j)\gamma(-x, -t)\} & = & (\sum_j b a_j z^j)\gamma(-x, -t) \\ \partial_{k\beta}\{(\sum_j a_j z^j)\gamma(-x, -t)\} & = & (\sum_j \partial_{k\beta}(a_j) z^j - \sum_j a_j E_{\beta\beta} z^{k+j})\gamma(-x, -t, z) \\ \partial\{(\sum_j a_j z^j)\gamma(-x, -t)\} & = & (\sum_j \partial_{k\beta}(a_j) z^j - \sum_j a_j z^{1+j})\gamma(-x, -t) \end{array}$$

In particular we see that ∂ acts invertible on M^*, so that the natural extension of these actions leads to an $M_n(R)[\partial,\partial^{-1})$-module structure on M^*, which is again free, since we have

$$\sum_j P_j(-\partial)^j \cdot \exp(-xz - \sum t_{i\alpha} z^i E_{\alpha\alpha}) = \{\sum_j P_j z^j\} \exp(-xz - \sum t_{i\alpha} z^i E_{\alpha\alpha}).$$

Now we have a bilinear pairing $\mathcal{R} : M \times M^* \to M_n(R)$ defined as follows: if $\varphi = \varphi(x,t,z) = (\sum_j a_j(x,t)z^j)\exp(xz + \sum t_{i\alpha} z^i E_{\alpha\alpha})$ belongs to M and $\psi = \psi(x,t,z) = (\sum_k b_k(x,t)z^k)\exp(-xz - \sum t_{i\alpha} z^i E_{\alpha\alpha})$ is an element of M^*, then we put

$$\begin{aligned}\mathcal{R}(\varphi,\psi) : &= \operatorname{Res}_z((\textstyle\sum_j a_j(x,t)z^j)(\sum b_k(x,t)^T z^k))\\ &= \textstyle\sum_{k\in\mathbb{Z}} a_{-k-1}(x,t) b_k(x,t)^T,\end{aligned}$$

where a^T for each a in $M_n(R)$ denotes the transposed of the matrix a. Clearly the defining sum in $\mathcal{R}$ is finite, hence belongs to $M_n(R)$. On the algebra $M_n(R)[\partial,\partial^{-1})$ we have a $\mathbb{C}$-linear anti-algebra morphism called "taking the adjoint". The adjoint of $P = \sum P_j \partial^j$ is given by

$$\begin{aligned}P^* &= \textstyle\sum_j (-\partial)^j P_j^T = \sum_j (-1)^j \sum_{k\geq 0} \begin{pmatrix} j \\ k \end{pmatrix} \partial^k(P_j^T)\partial^{j-k}\\ &= \textstyle\sum_l \left\{\sum_{k\geq 0}(-1)^{l+k}\begin{pmatrix} l+k \\ k\end{pmatrix}\partial^k(P_{l+k}^T)\right\}\partial^l.\end{aligned}$$

The following theorem gives an important connection between the bilinear form $\mathcal{R}$ and taking the adjoint. Its proof can be found in [HL00]

Theorem 4.1. *Let $\varphi(x,t,z) = P(x,t,\partial)\cdot\exp(xz + \sum t_{i\alpha} z^i E_{\alpha\alpha}) \in M$ and $\psi(t,z) = Q(x,t,\partial)\cdot \exp(-xz - \sum t_{i\alpha} z^i E_{\alpha\alpha}) \in M^*$, then*

$$\begin{aligned}(P(x,t,\partial)Q(x,t,\partial)^*)_- &= -\textstyle\sum_{n=0}^{\infty} \mathcal{R}(\varphi(x,t,z), \partial^n(\psi(x,t,z)))(-\partial)^{-n-1}\\ &=: \mathcal{R}(\varphi(x,t,z), \partial^{-1}\circ\psi(x,t,z))\end{aligned} \tag{35}$$

Like we have in M the notion of oscillating matrix function of type δ, we have in M^* that of dual oscillating matrix function of type $-\delta$. Consider thereto the matrix

$$R^-(\delta;z) := \sum_{i=1}^n \varepsilon(\varepsilon_i,\delta)E_{ii}(-z)^{-(\varepsilon_i|\delta)} = R^+(-\delta;-z) \tag{36}$$

$$= diag(\varepsilon(\varepsilon_1,\delta)(-1)^{-(\varepsilon_1|\delta)},\ldots,\varepsilon(\varepsilon_n,\delta)(-1)^{-(\varepsilon_n|\delta)})z^{-\delta}. \tag{37}$$

Note that we have in all points of the lattice that

$$R^-(\delta;\partial) = (R^+(\delta;\partial)^{-1})^*. \tag{38}$$

Then a *dual oscilating function of type* $-\delta$ is by definition an element ψ^- in M^* of the form

$$\begin{aligned}\psi^-(\delta;x,t,z) &= Q(\delta;x,t,z)\cdot R^-(\delta;\partial)\cdot\exp(-xz - \textstyle\sum t_{i\alpha} z^i E_{\alpha\alpha})\\ &= \left\{(1+\textstyle\sum_{j<0} g_j z^j)\right\} R^-(\delta;z)\exp(-xz - \sum t_{i\alpha} z^i E_{\alpha\alpha})\end{aligned}$$

If $\varphi^+(\delta; x, t, \partial) = P^+(\delta; x, t, \partial)R^+(\delta; \partial) \cdot \exp(xz + \sum t_{i\alpha} z^i E_{\alpha\alpha})$ in M is an oscillating matrix function of type δ, then

$$\varphi^-(\delta; x, t, \partial) = (P^+(\delta; x, t, \partial)^*)^{-1}(R^+(\delta; \partial)^{-1})^* \cdot \exp(-xz - \sum t_{i\alpha} z^i E_{\alpha\alpha}) \tag{39}$$

$$= P^-(\delta; x, t, \partial) \cdot R^-(\delta; \partial) \cdot \exp(-xz - \sum t_{i\alpha} z^i E_{\alpha\alpha}) \tag{40}$$

is clearly a dual oscillating matrix function of type $-\delta$. It is called the adjoint of φ. Since $(P(x, t, \partial)((P(x, t, \partial)^*)^{-1})^*)_- = 0$, the foregoing theorem shows that for all $n \geq 0$

$$\mathcal{R}(\varphi, \partial^n(\varphi^*)) = 0. \tag{41}$$

This property even characterizes φ^* among the dual oscillating matrix functions of type $-\delta$. For, if $\psi = Q(x, t, \partial)R^-(-\delta; \partial) \cdot \exp(-xz - \sum t_{i\alpha} z^i E_{\alpha\alpha}) \in M^*$ is such a function satisfying (41) with φ^* replaced by ψ, then we have according to the theorem

$$PR^+(\delta; \partial)(R^-(\delta, ; \partial)^{-1})^* Q^* = PQ^* = \partial^0 + (PQ^*)_- = 1.$$

In other words, $Q = (P^*)^{-1}$ and ψ is the adjoint of φ. We will use this criterion later on and therefore we resume it in a

Corollary 4.1. *Let φ be an oscillating matrix function of type δ and let ψ be a dual oscillating matrix function of type $-\delta$. Then ψ is the adjoint of φ if and only if it satisfies*

$$\mathcal{R}(\varphi(x, t, z), \partial^n(\psi(x, t, z))) = 0 \text{ for all } n \geq 0.$$

If φ^+ is a wave function of the n-component KP-hierarchy, then its adjoint φ^- is called a dual wave function of the n-component KP-hierarchy. It satisfies a set of linear equations similar to (17), the so-called adjoint system, see [HL00],

$$\begin{aligned} L(\delta; x, t, \partial)^* \varphi^-(\delta; x, t, z) &= z\varphi^-(\delta; x, t, z), \\ U_\alpha(\delta; x, t, \partial)^* \varphi^-(\delta; x, t, z) &= \varphi^-(\delta; x, t, z)E_{\alpha,\alpha}, \\ \frac{\partial \varphi^-(\delta; x, t, z)}{\partial t_{k\alpha}} &= -(L(\delta; x, t, \partial)^{*k} U_\alpha(\delta; x, t, \partial)^*)_+ \varphi^-(\delta; x, t, z). \end{aligned} \tag{42}$$

The bilinear form $\mathcal{R}$ plays also an important role at the interpretation of the famous bilinear identities for an oscillating function ψ^+ and and its dual, (see [DJKM]):

$$Res_z \psi^+(\delta; x, t, z)\psi^-(\delta; y, s, z)^T = 0. \tag{43}$$

First we have a look at the formal product $\psi(x, t, z)\phi(x, s, z)^T$, where $\psi(x, t, z) = \{\sum_k a_k(x, t)z^k\}\gamma(x, t, z) \in M$ and $\phi(y, t, z) = \{\sum_m b_m(y, s)z^m\}\gamma(-y, -t, z) \in M^*$. If we write $e^{xz + \sum t_{i\alpha} z^i E_{\alpha\alpha}} = \sum_{r=0}^\infty p_r(x, (t_{i\alpha}))z^r$, then this formal product becomes

$$(\sum_k a_k(x, t)z^k)(\sum_{r=0}^\infty p_r(x - y, (t_{i\alpha} - s_{i\alpha}))z^r)(\sum_m b_m(y, s)^T z^m) \tag{44}$$

Hence the equation (43) for $\psi = \psi^+$ and $\phi = \psi^-$ boils formally down to

$$\sum_{r=0}^\infty \sum_k a_k(x, t)p_r(x - y, (t_{i\alpha} - s_{i\alpha}))b_{-r-1-k}(y, s)^T = 0,$$

for all relevant $t = (t_{i\alpha})$ and $s = (s_{i\alpha})$. In order to avoid convergence considerations, we look at a few differential consequences of these relations that always exist. If one differentiates the equation namely with respect to x or some of the parameters $(t_{i\alpha})$ or $(s_{i\alpha})$ and substitutes next $t_{i\alpha} = s_{i\alpha}$ for all $i \geq 1$, then one ends up with finite expressions in the coefficients of ψ and ψ^*. Thus we get for example for all $k \geq 0$, $i \geq 1$, $1 \leq \alpha \leq n$, and all $m \geq 0$ the relations

$$\mathcal{R}(\partial^k(\psi^+(\delta; x, t, z)), \partial^m(\psi^-(\delta; x, t, z))) = 0, \tag{45}$$

$$\mathcal{R}(\partial_{i\alpha}(\psi^+(\delta; x, t, z)), \partial^m(\psi^-(\delta; x, t, z))) = 0. \tag{46}$$

Note that this first equation can also be obtained by applying ∂ several times to the relations (41). Assume now that an oscillating matrix function ψ^+ of type δ and its dual ψ^- satisfy the equations (45) and (46). Let $L = P\partial P^{-1}$ and $U_\alpha = PE_{\alpha\alpha}P^{-1}$ be the operators associated with ψ^+. From the form of the action of the $\partial_{i\alpha}$ on ψ^+ one sees that there exists for each $i \geq 1$ and all $\alpha, 1 \leq \alpha \leq n$, an differential operator $Q_{i\alpha}$ in $R[\partial]$ of degree i such that

$$\partial_{i\alpha}(\psi^+) - Q_{i\alpha}(\psi^+) = G_{i\alpha}R^+(\delta; \partial)e^{xz + \sum_i t_i z^i}, \text{ with } G_{i\alpha} \in R[\partial, \partial^{-1}) \text{ of degree } < 0.$$

Because of the relations in (45) and (46) we have then for all $m \geq 0$ that

$$\mathcal{R}(\partial_{i\alpha}(\psi(x, t, z)) - (Q_{i\alpha})(\psi^+(x, t, z)), \partial^m(\psi^-(x, t, z))) = 0. \tag{47}$$

According to theorem 4.1, we get then that $(G_{i\alpha}P^{-1})_- = G_{i\alpha}\partial^l P^{-1} = 0$. Thus we have obtained that $\partial_{i\alpha}(\psi(x, t, z)) - (Q_{i\alpha})(\psi(x, t, z)) = 0$. From proposition 3.1 follows for all $i \geq 1$ and all α that the operator $Q_{i\alpha}$ is equal to $(L^i U_\alpha)_+$. Hence ψ^+ is a wave function of the n-component KP-hierarchy. Reversely, if ψ^+ is a wave function of the the n-component KP-hierarchy, then the first equation in (45) holds and since $\partial_{i\alpha}$ acts on ψ^+ as $(L^i U_\alpha)_+$ also the second relation holds. Thus we have found the following characterization of the wave functions in M

Proposition 4.1. *An oscillating matrix function ψ^+ of type δ is a wave function of the n-component KP-hierarchy if and only if ψ^+ and its dual ψ^- satisfy the equations (45) and (46).*

A consequence of this proposition is that wave functions of the n-component KP-hierarchy are characterized by the following set of equations

$$\Re(\Delta_1(\psi^+(x, t, z)), \Delta_2(\psi^-(x, t, z))) = 0, \tag{48}$$

where Δ_1 and Δ_2 are arbitrary finite products of the $\{\partial_{i\alpha}\}$ and ∂. These equations are the algebraic version of the equations (43).

5 The Sato Grassmannian

In this section we describe the Sato Grassmannian, from which one can construct matrix wave functions of the n-component KP-hierarchy. It is a vector form of the line set out in [Sh]. Since the rows of these functions consist of infinite series in z and z^{-1} with coefficients from $\mathbb{C}^n$, it is not so strange that the basic manifold for the relevant subspaces consists of a class of series in z and z^{-1}. Consider the spaces

$$\begin{aligned} H_- &= (z^{-1}\mathbb{C}[[z^{-1}]])^n = \{\textstyle\sum_{j=1}^{\infty} a_j z^{-j} | a_j \in \mathbb{C}^n\} \quad \text{and} \\ H_+ &= (\mathbb{C}[z])^n = \{\textstyle\sum_{i=0}^{m} b_i z^i | b_i \in \mathbb{C}^n\}. \end{aligned}$$

Hence $H = H_+ \oplus H_-$ is equal to $(\mathbb{C}((z)))^n$, where $\mathbb{C}((z))$ is the quotient field of $\mathbb{C}[[z^{-1}]]$. First we fix some notations in the space H. For l, $0 \le l \le n-1$, let f_l be the vector in $\mathbb{C}^n$ with a one at the $l+1$-th place and elsewhere zeros. Then we denote for all $m \in \mathbb{Z}$ and all l, $0 \le l \le n-1$, the vector $f_l z^m$ by e_{l+mn}. Inside H we consider the subspaces $H_k, k \in \mathbb{Z}$ given by

$$H_k := \{\sum_{r \ge k} a_r e_r, \text{ with } a_r \in \mathbb{C} \text{ and } \sum_{r \ge k} a_r e_r \in H\}. \tag{49}$$

For each $k \in \mathbb{Z}$ let $p_k : H \to H_k$ be the projection

$$p_k(\sum a_j e_j) = \sum_{j \ge k} a_j e_k.$$

Then the Grassmann manifold of Sato consists of all subspaces of H that are of a size comparable to H_+. More precisely, it is given by

$$Gr(H) = \left\{ W \,\middle|\, \begin{array}{l} W \subset H, p_0 : W \to H_+ \text{ has a finite} \\ \text{dimensional kernel and cokernel} \end{array} \right\}.$$

The space $Gr(H)$ has a subdivision into different components according to the index of $p_0|_W$ that is defined by

$$\text{ind } (p_0|_W) = \dim(\text{Ker}(p_0|_W)) - \dim(\text{Coker}(p_0|_W)).$$

We denote these components as follows

$$Gr^{(k)}(H) = \{W | W \in Gr(H), \text{ind}(p_0|_W) = -k\}.$$

Clearly, the subspace H_k belongs to $Gr^{(k)}(H)$ and one easily verifies that the component $Gr^{(k)}(H)$ can also be described as

$$Gr^{(k)}(H) = \{W | W \in Gr(H), \text{ind}(p_k|_W) = 0\}.$$

Also all subspaces in $Gr(H)$ that project bijectively onto H_k, i.e. all W belonging to the "big cell"

$$O^{(k)} = \left\{ W \,\middle|\, \begin{array}{l} W \in Gr(H), w \mapsto p_k(w) \text{ is a} \\ \text{bijection: } W \to H_k \end{array} \right\},$$

belong to $Gr^{(k)}(H)$. To have a description of all planes in $Gr^{(k)}(H)$, consider for each k in $\mathbb{Z}$ the collection of sequences

$$\mathcal{S}(k) = \left\{ (s_i) \,\middle|\, \begin{array}{l} i \in \mathbb{Z}, i \ge k, s_i \in \mathbb{Z}, s_{i+1} > s_i \text{ and } s_l = l \\ \text{for } l \text{ sufficiently large} \end{array} \right\}.$$

For $\sigma = (s_i)$ in $\mathcal{S}(k)$, one has the subspace $H(\sigma)$ in $Gr^{(k)}(H)$ given by

$$H(\sigma) = \text{Span } \{e_{s_i} | \sigma = (s_i)\}.$$

Let $p(\sigma) : H \to H(\sigma)$ be the projection

$$p(\sigma)(\sum \alpha_i e_i) = \sum_{s_i,\ \sigma = (s_i)} \alpha_{s_i} e_{s_i}.$$

Since each plane in $Gr(H)$ has a basis of elements with different leading coefficients, one sees that there exists for each $W \in Gr^{(k)}(H)$ a σ in $\mathcal{S}(k)$ such that the projection $p(\sigma) : W \to H(\sigma)$ is a bijection. As each $H(\sigma), \sigma = (s_j) \in \mathcal{S}(k)$, is the image of the embedding $i_\sigma : H_k \to H$ given by

$$i_\sigma(e_j) = e_{s_j} \quad , \quad j \geq k,$$

one sees that the planes in $Gr^{(k)}(H)$ can be described as follows

Proposition 5.1. *Each $W \in Gr^{(k)}(H)$ is the image of an embedding $w : H_k \to H$ such that, if $w(e_j) = \sum_{i \geq k} w_{ij} e_i$ for all $j \geq k$, the upper part w_+ of the matrix $[w] = (w_{ij})$ has the form*

$$w_+ = \begin{pmatrix} \ddots & & \vdots \\ & w_{k+1,k+1} & w_{k+1,k} \\ \cdots & w_{k,k+1} & w_{k,k} \end{pmatrix} = \begin{pmatrix} Id & 0 \\ B & A \end{pmatrix},$$

where A is a square matrix of finite size. Reversely, for every such embedding w, its image belongs to $Gr^{(k)}(H)$. In particular $w(H_k)$ belongs to the big cell iff $\det(A)$ is non zero.

Notation: We denote the collection of embeddings $w : H_k \to H$ that occur in the proposition by $\mathcal{P}_k$.
On the space H we have a bilinear form B that plays a role in the sequel. Namely, if $f = \sum_j a_j z^j$ and $g = \sum_j b_j z^j$ are in H, then we define

$$B(f, g) = Res_z(f(z)g(z)^T) = \sum_j a_j b^T_{-j-1}, \tag{50}$$

where we have written the elements of $\mathbb{C}^n$ as rows and A^T denotes the transposed of a matrix A. For W in $Gr(H)$, let $W^\perp$ be the orthocomplement of W in H w.r.t.this form B. With the above given description of spaces in $Gr(H)$ one verifies that $W^\perp$ also belongs to $Gr(H)$.

Our next step will be to show how the subspaces from $Gr(H)$ occur as the span of the Laurent coefficients of certain oscillating functions. First we describe the ring R and its derivations $\partial_{n\alpha}$ and ∂. Consider the ring $\mathbb{C}[[x, t]]$ of formal powerseries in x and $t = (t_{i\alpha})$. Let θ be an element in $\mathbb{C}[[x, t]]$ of the form

$$\theta(x, 0) = x^N + \sum_{j>N} a_j x^j. \tag{51}$$

In the sequel the ring R will always be the localization of $\mathbb{C}[[x, t]]$ w.r.t. the multiplicative subset $S_\theta = \{\theta^m | m \geq 0\}$ for some θ in $\mathbb{C}[[x, t]]$ of the form (51). On R we take $\partial_{i\alpha} = \dfrac{\partial}{\partial t_{i\alpha}}$ and $\partial = \dfrac{\partial}{\partial x}$.
Consider a matrix wavefunction ψ^+ of the n-component KP-hierarchy and its dual ψ^- that have the form

$$\psi^+(\delta; x, t, z) = \{\sum_{j \leq 0} a_j(x, t) z^j\} R^+(\delta; z) \exp(xz + \sum t_{i\alpha} z^i E_{\alpha\alpha}) \tag{52}$$

$$\psi^-(\delta; x, s, z) = \{\sum_{m \leq 0} b_m(x, t) z^m\} R^-(\delta; z) \exp(-xz - \sum t_{i\alpha} z^i E_{\alpha\alpha}). \tag{53}$$

The class of wavefunctions we will consider in this paper that satisfies the condition that there is an θ in $\mathbb{C}[[x,t]]$ of the form (51) such that for all $m \le 0$ and all $j \le 0$

$$\theta(x,t)a_j(x,t) \in M_n(\mathbb{C}[[x,t]]) \text{ and } \theta(x,t)b_m(x,t) \in M_n(\mathbb{C}[[x,t]]). \tag{54}$$

These matrix wavefunctions are called *regularizable*. For regularizable wavefunctions the Laurent series in x of ψ and ψ^* have the form

$$\psi^+(\delta;x,t,z) = \sum_{j\ge -N} w_j(t,z)x^j \exp(xz + \sum t_{i\alpha}z^i E_{\alpha\alpha}), \tag{55}$$

$$\text{where } w_j(t,z) = \sum_{l=-\infty}^{N_1} v_l z^l, \text{ with } v_l \in M_n(\mathbb{C}[[t]]), \tag{56}$$

$$\psi^-(x,t,z) = \sum_{j\ge -N} w_j^*(t,z)x^j \exp(-xz - \sum t_{i\alpha}z^i E_{\alpha\alpha}), \tag{57}$$

$$\text{where } w_j^*(t,z) = \sum_{l=-\infty}^{N_2} v_l^* z^l, \text{ ; with } v_l^* \in M_n(\mathbb{C}[[t]]). \tag{58}$$

It is not hard to show that both spaces

$$W = Span\{ \text{ rows of } w_j(0,z),\ j \ge -N\} \text{ and } W^* = Span\{ \text{ rows of } w_j^*(0,z),\ j \ge -N\}$$

belong to $Gr(H)$. Following ideas of Sato, one can show, see [DJKM], that the space W even determines ψ, for there holds

Proposition 5.2. *The map B_δ that associates to a regularizable wavefunction ψ of type δ of the n-component KP-hierarchy the span of the rows of the coefficients in $t = 0$ of the Laurent series of ψ^+ in x is an injection from the class of matrix wavefunctions of type δ into $Gr(H)$. All planes in $Gr(H)$ occur in the image of this map for a certain type δ. The same properties hold for the map B_δ^* that associates to a regularizable dual wavefunction ψ^- of type $-\delta$ of the n-component KP-hierarchy the span of the coefficients in $t = 0$ of the Laurent series of ψ^- in x.*

It might happen that there correspond to a certain plane, matrix wavefunctions of various types, see [HP]. The properties of the index imply that a plane $W \in Gr^{(k)}(H)$ can only belong to the image of B_δ, if $\delta \in k\epsilon_1 + \Lambda$, where $\epsilon = \epsilon_1 + \ldots + \epsilon_n$, and $\Lambda = \{\gamma \in \mathbb{Z}^n | \ (\epsilon|\gamma) = 0\}$. Note that Λ is nothing but the root lattice of $s\ell_n(\mathbb{C})$. The wavefunctions that satisfy the conditions in (54) for $\theta = 1$ correspond to the union of the big cells. For each δ in the lattice and each $W \in Gr(H)$ in the image of B_δ, we write the matrix wavefunction of type δ corresponding to W and its dual matrix wavefunction as

$$\psi_W^+(\delta) = \hat{\psi}_W^+(\delta)R^+(\delta,z)\exp(xz + \sum t_{i\alpha}z^i E_{\alpha\alpha}) \text{ resp.} \tag{59}$$

$$\psi_W^-(\delta) = \hat{\psi}_W^-(\delta)R^-(\delta,z)\exp(-xz - \sum t_{i\alpha}z^i E_{\alpha\alpha}). \tag{60}$$

Now that we have this link between matrix wavefunctions and the Grassmann manifold, we can also give a geometric description of the dual wavefunction of $\psi_W(\delta)^+$. From the characterizing properties in (45), follows

Proposition 5.3. *Let W belong to the image of B_δ and let $\tilde{W}$ be a subpace in $Gr(H)$. Then $\tilde{W}$ is the space W^* corresponding to the dual wavefunction $\psi^-_W(\delta)$, if and only if $\tilde{W} = W^\perp$ with $W^\perp$ the orthocomplement of W w.r.t. the bilinear form B on H.*

6 The tau-functions

In the foregoing section we saw how one could associate to a regularizable wavefunction of the KP-hierarchy a plane in $Gr(H)$. The present section is devoted to an explicit description of the inverse of the map B_δ, i.e. we will show how to build a wavefunction of type z^δ, starting from an element in the image of B_δ. Before giving the construction, we need to introduce some notations for operators that act on H. For each pair (i,j), with $i \neq j$, $1 \leq i \leq n$ and $1 \leq j \leq n$, we consider the element z^κ with $\kappa_i = 1$, $\kappa_j = -1$ and all $\kappa_l = 0$, for l different from i and j. Let $\Delta_{ij} \in \mathrm{End}(H)$ be defined by $\Delta_{ij}(h) = hz^\kappa$. In $\mathrm{End}(H)$, we have the basic operators $E_{(k,l)}$, with k and $l \in \mathbb{Z}$, defined by

$$E_{(k,l)}\Big(\sum_{r\in\mathbb{Z}} \alpha_r e_r\Big) = \alpha_l e_k. \tag{61}$$

In terms of these operators we have that

$$\Delta_{ij} = \sum_{l\in\mathbb{Z}} \left(\sum_{\substack{0\leq k\leq n-1 \\ k\neq i-1 \\ k\neq j-1}} E_{(k+ln,k+ln)} \; + \; E_{(i-1+n(l+1),i-1+nl)} \; + \; E_{(j-1+n(l-1),j-1+nl)} \right).$$

Now we have that Δ_{ij} decomposes w.r.t. $H = H_+ \oplus H_-$ as

$$\Delta_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix}.$$

It is a straightforward computation that the kernel of a_{ij} is equal to $\mathbb{C}f_{j-1}z^0$ and that the cokernel of this operator is isomorphic to $\mathbb{C}f_{i-1}z^0$. Hence if we define the operator q_{ij} by $q_{ij} := a_{ij} + E_{(i-1,j-1)}$, then one verifies directly that this defines a bijective endomorphism of H. Moreover there holds

$$a_{ij}q_{ij}^{-1} = Id - E_{(i-1,j-1)}q_{ij}^{-1} = Id - E_{(i-1,i-1)}. \tag{62}$$

Now, let $WR^+(\delta;z)^{-1} \in Gr^{(0)}(H)$ be the image of an embedding $w : H_+ \to H$ in $\mathcal{P}_0$. If $w(e_j) = \sum_{i\in\mathbb{Z}} w_{ij}e_i$ for all $j \geq 0$, then the matrix of w decomposes w.r.t. $H = H_+ \oplus H_-$ in the components $w_+ = (w_{ij})$, $i \geq 0$ and $j \geq 0$ and $w_- = (w_{ij})$, $i \geq 0$ and $j < 0$. Since w belongs to $\mathcal{P}_0$, we may assume that the $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$-matrix w_+ has w.r.t. $H_+ = H_l \oplus Span(e_m,\ 0 \leq m < l)$ for sufficiently large l the form

$$w_+ = \begin{pmatrix} Id & 0 \\ B & A \end{pmatrix}. \tag{63}$$

From the relation (62), one concludes that also the embedding $w_{(i,j)} := \Delta_{ij}wq_{ij}^{-1}$ has for a sufficiently large l a decomposition of the form (63).

Though the multiplication of elements in H with the exponential factor $\exp(xz + \sum t_{i\alpha} z^i E_{\alpha\alpha})$ brings them outside of H, we will nevertheless consider the matrix with respect to this operation. First we write

$$\exp(xz + \sum t_{i\alpha} z^i E_{\alpha\alpha}) = \sum_{i=0}^{\infty} p_l(x,t) z^l,$$

where p_l is a diagonal matrix in $M_n(\mathbb{C}[x,t])$, whose (i,i)-th entry is the homogeneous polynomial $p_l^{(i)}$ of degree l. Here we have put the degree of x equal to one and that of $t_{i\alpha}$ equal to i. Then the corresponding $\mathbb{Z} \times \mathbb{Z}$-matrix $[\gamma] = (\gamma_{(i,j)})$ with coefficients in $\mathbb{C}[[x,t]]$ is given by

$$\gamma_{(i-1+kn, i-1+mn)} = p_{k-m}^{(i)}, \quad k \geq m, \text{ and } \gamma_{(i,j)} = 0 \text{ for the remaining pairs } (i,j).$$

Also the matrix $[\gamma]$ we decompose w.r.t. $H = H_+ \oplus H_-$ as

$$[\gamma] = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}.$$

Clearly, the $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$-matrix α is invertible within the $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$-matrices with coefficients from $\mathbb{C}[[x,t]]$. One verifies directly that the product $[\gamma][w]\alpha^{-1}$ is a well-defined $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$-matrix with coefficients in $\mathbb{C}[[x,t]]$. From the special form of w_+ one sees directly that the $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$-matrix

$$([\gamma][w]\alpha^{-1})_+ = \alpha w_+ \alpha^{-1} + \beta w_- \alpha^{-1}$$

is the sum of an invertible $\mathbb{Z}_{\geq k} \times \mathbb{Z}_{\geq k}$-matrix with coefficients from $\mathbb{C}[[x,t]]$ and one with a finite dimensional range, hence it has a well-defined determinant belonging to $\mathbb{C}[[x,t]]$. The same holds if one replaces w by $w_{(i,j)}$, since the crucial property is (63) and that remains preserved. We define now the τ-function corresponding to w by

$$\tau_w(\delta; x, t) := \det(\alpha w_+ \alpha^{-1} + \beta w_- \alpha^{-1}). \tag{64}$$

The remark after proposition 5.2 implies that we merely can consider this function for δ of the form $\delta = k\epsilon_1 + \Lambda$. The fact if the function $\tau_w(\delta)$ is nonzero or not, determines if the space W belongs to the image of the map B_δ. If so, then one shows by using the fact that there exists for W a $H(\sigma)$ with σ in $\mathcal{S}(k)$ such that W projects bijectively onto $H(\sigma)$, that $\tau_w(\delta)$ satisfies condition (51). By following the same line of reasoning as in ([HP93]) one can express the matrixcoefficients of the wavefunction and its dual in the τ-functions, see also ([DJKM]),

Theorem 6.1. *Let $w \in \mathcal{P}_0$ be an embedding of H_+ into H with image $WR^+(\delta; z)^{-1}$ and let the corresponding τ-function $\tau_w(\delta)$ be nonzero. Then we have the following formulae for the (i,j)-th coefficient of $\hat{\psi}_W^+(\delta; x, t, z)$ and $(\psi_W^-(\delta; x, t, z)$*

$$\hat{\psi}_W^+(\delta; x, t, z)_{ii} = \frac{\tau_w(\delta; x, (t_{k\alpha} - \delta_{i\alpha} \frac{1}{kz^k}))}{\tau_w(\delta; x, t)},$$

$$\hat{\psi}_W^+(\delta; x, t, z)_{ij} = z^{-1} \frac{\tau_{w_{(i,j)}}(\delta; x, (t_{k\alpha} - \delta_{i\alpha} \frac{1}{kz^k}))}{\tau_w(\delta; x, t)} \text{ for } i \neq j,$$

$$\hat{\psi}_W^-(\delta; x, t, z)_{ii} = \frac{\tau_w(\delta; x, (t_{k\alpha} + \delta_{i\alpha} \frac{1}{kz^k}))}{\tau_w(\delta; x, t)},$$

$$\hat{\psi}^{-}_{W}(\delta; x, t, z)_{ij} = z^{-1} \frac{\tau_{w_{(i,j)}}(\delta; x, (t_{k\alpha} + \delta_{i\alpha} \frac{1}{kz^k}))}{\tau_w(\delta; x, t)} \text{ for } i \neq j.$$

Let $\tau = \{\tau_{ij}(\delta) | 1 \leq i \leq n, \ 1 \leq j \leq n\}$ be a collection of elements in $\mathbb{C}[[x, t]]$ such that $\tau_{ii}(\delta) = \tau_{11}(\delta)$ for all $i, 1 \leq i \leq n$, and assume that $\tau_{11}(\delta)$ has the form (51). Then we can define an oscillating matrix function $\phi_\tau(\delta) = \hat{\phi}_\tau(\delta) R^+(\delta; z)\gamma(x, t, z)$ of type δ by the formulae

$$\hat{\phi}_\tau(\delta; x, t, z)_{ii} = \frac{\tau_{ii}(\delta; x, (t_{k\alpha} - \delta_{i\alpha} \frac{1}{kz^k}))}{\tau_{11}(\delta; x, t)},$$

$$\hat{\phi}_\tau(\delta; x, t, z)_{ij} = z^{-1} \frac{\tau_{ij}(\delta; x, (t_{k\alpha} - \delta_{i\alpha} \frac{1}{kz^k}))}{\tau_{11}(\delta; x, t)}, \text{ for } i \neq j.$$

As we have shown in theorem 4.1 this ϕ_τ is a wavefunction of the KP-hierarchy if and only if it satisfies the equations (45) and (46). By expressing these equations in $(\tau_{ij}(\delta))$, one ends up with a collection of so-called Hirota bilinear identities for the $(\tau_{ij}(\delta))$.

Remark 6.1. *The dependence of the function τ_w of the embedding is just mild: another embedding with the same image changes the function τ_w only into a nonzero multiple. This constant is not relevant in a lot of questions, see e.g. the formulae for the coefficients of the wavefunctions in theorem 6.1. Therefore we write $\tau_W(\delta; x, t)$ in the sequel for the τ-function associated with W and δ.*

7 The construction of the transformations

In this section we shall use the formal calculus of vertex operators as e.g. given in [K] to obtain the elementary Darboux transformations. We will start this section with a recollection of some of the results of the previous sections that we will need here.

Let $\delta \in \mathbb{Z}^n$ be such that $(\delta|\epsilon) = k$ and assume $W \in Gr^{(k)}(H)$ belongs to the image of B_δ. Then we have the following expression for the corresponding wave functions and adjoint wave functions

$$\psi^{\pm}_{W}(\delta; x, t, z) = P^{\pm}(\delta; x, t, \partial) R(\pm\delta; \partial)\gamma(\pm x, \pm t, z). \tag{65}$$

The wave function and adjoint wave functions are taken to be zero if $(\delta|\epsilon) \neq k$. The associated solutions of the n-component KP-hierarchy are

$$\begin{aligned} L(\delta; x, t, \partial) &= P^+(\delta; x, t, \partial)\partial P^+(\delta; x, t, \partial)^{-1}, \\ U_\alpha(\delta; x, t, \partial) &= P^+(\delta; x, t, \partial) E_{\alpha\alpha} P^+(\delta; x, t, \partial)^{-1}. \end{aligned} \tag{66}$$

By using the fundamental theorem 4.1 one can prove the following

Proposition 7.1. *Let $\gamma, \delta \in k\epsilon_1 + \Lambda$, then $P^-(\delta; x, t, \partial) = P^+(\delta; x, t, \partial)^{*-1}$,*

$$(P^+(\delta; t, x, \partial) R(\delta - \gamma, \partial) P^-(\gamma; x, t, \partial)^*)_- = 0 \tag{67}$$

and they satisfy the Sato-Wilson equations:

$$\frac{\partial P^{\pm}(\delta; x, t, \partial)}{\partial t_{k\alpha}} = \mp (P^{\pm}(\delta; x, t, \partial) E_{\alpha\alpha}(\pm\partial)^k P^{\pm}(\delta; x, t, \partial)^{-1})_- P^{\pm}(\delta; x, t, \partial). \tag{68}$$

The Sato-Wilson equations are related to the linear equations (16) and (17) for ψ_W^+ and the adjoint system (42) for ψ_W^-. The simplest equations in these linear systems are

$$\frac{\partial\psi_W^\pm(\delta;x,t,z)}{\partial t_{1\alpha}} = \left(\partial E_{\alpha\alpha} + \left[W^\pm(\delta;x,t),E_{\alpha\alpha}\right]\right)\psi_W^\pm(\delta;x,t,z), \tag{69}$$

where

$$W^\pm(\delta;x,t) = \mathrm{Res}_\partial P^\pm(\delta;x,t,\partial). \tag{70}$$

Here $\mathrm{Res}_\partial \sum_j f_j\partial^j = f_{-1}$. As we saw at the end of section 4 these wave functions and their adjoints satisfy the bilinear identity:

$$\mathrm{Res}_z \psi_W^-(\delta;x,t,z)\psi_W^+(\gamma;x',t',z)^T = 0 \text{ for all } \gamma,\delta\in k\epsilon_1+\Lambda. \tag{71}$$

With the help of the expressions found in theorem 6.1 we can write the bilinear identity (71) as a tower of equations for the collection of tau functions $\tau_W(\delta;x,t)$ $(\{\delta\in k\epsilon_1+\Lambda|\tau_W(\delta;x,t)\neq 0\})$. Decompose the wave function and its adjoint as

$$\psi_W^\pm(\delta;x,t,z) = (\psi_W^\pm(\delta;x,t,z)_{ij})_{i,j=1}^n. \tag{72}$$

Then the formulae in theorem 4.1 imply

$$\begin{aligned}\psi_W^\pm(\delta;x,t,z)_{ij} &= \frac{\varepsilon(\epsilon_j,\delta+\epsilon_i)}{\tau_W(\delta;x,t)} z^{(\epsilon_j|\pm\delta+\epsilon_i-\epsilon_j)} \\ &\times \exp(\pm xz\pm\sum_{k=1}^\infty t_{kj}z^k)\exp(\mp\sum_{k=1}^\infty \frac{\partial}{\partial t_{kj}}\frac{z^{-k}}{k})\tau_W(\delta\pm(\epsilon_i-\epsilon_j);x,t)\end{aligned} \tag{73}$$

where $\varepsilon:\ \mathbb{Z}^n\times\mathbb{Z}^n\to\{\pm1\}$ is the bimultiplicative function as defined in (28). It is now straightforward to check that

$$W^\pm(\delta;x,t)_{ij} = \begin{cases} \pm\varepsilon_{ji}\frac{\tau(\delta\pm\alpha_{ij};x,t)}{\tau(\delta;x,t)} & \text{if } i\neq j,\\ -\frac{\partial\log(\tau(\delta;x,t))}{\partial t_{1i}} & \text{if } i=j.\end{cases} \tag{74}$$

Consider the vector space $\mathbb{C}[\mathbb{Z}^n]$ with basis e^γ, $\gamma\in\mathbb{Z}^n$, and the following twisted group algebra product:

$$e^\alpha e^\beta = \varepsilon(\alpha,\beta)e^{\alpha+\beta}. \tag{75}$$

Let $B = R\otimes_{\mathbb{C}}\mathbb{C}[\mathbb{Z}^n]$ be the tensor product of algebras. Define vertex operators on B

$$\phi^{\pm(j)}(z) = \sum_{i\in\frac12+\mathbb{Z}}\phi_i^{\pm(j)}z^{-i-\frac12} = e^{\pm\epsilon_j}z^{\pm\epsilon_j}\exp(\pm xz\pm\sum_{k=1}^\infty t_{kj})\exp(\mp\sum_{k=1}^\infty\frac{\partial}{\partial t_{kj}}\frac{z^{-k}}{k}), \tag{76}$$

then (see e.g. [K], [KvdL])

Lemma 7.1. *Let $\delta(y-z) = y^{-1}\sum_{k\in\mathbb{Z}}\left(\frac{y}{z}\right)^k$, then the operators $\phi^{\pm(i)}(z)$ satisfy the following anti-commutation relations:*

$$\begin{aligned}&\phi^{\pm(i)}(y)\phi^{\pm(j)}(z)+\phi^{\pm(j)}(z)\phi^{\pm(i)}(y) = 0,\\ &\phi^{\pm(i)}(y)\phi^{\mp(j)}(z)+\phi^{\mp(j)}(z)\phi^{\pm(i)}(y) = \delta_{ij}\delta(y-z).\end{aligned}$$

Let $W \in Gr^{(k)}(H)$ and $\tau_W(x,t) = \sum_{\delta\in k\epsilon_1+\Lambda} \tau_W(\delta;x,t)e^{\delta}$ be a KP tau-function, then we deduce from (71) that $\tau_W(x,t)$ satisfies the bilinear identity,

$$\mathrm{Res}_z \sum_{j=1}^{n} \phi^{+(j)}(z)\tau_W(x,t) \otimes \phi^{-(j)}(z)\tau_W(x,t) = 0. \tag{77}$$

Now let $f(z) \in H$, i.e.,

$$f(z) = (f_1(z), f_2(z), \ldots, f_n(z)) \quad \text{with } f_i(z) = \sum_{k=-\infty}^{N_i} f_{ik} z^k. \tag{78}$$

We let the following operator act on the bilinear identity (77) (where we assume that $\lambda =$ either $+$ or $-$)

$$\sum_{i=1}^{n} f_i(w)\phi^{\lambda(i)}(w) \otimes 1,$$

one thus obtains using Lemma 7.1:

$$\begin{aligned}
0 =& \left(\sum_{i=1}^{n} f_i(w)\phi^{\lambda(i)}(w) \otimes 1\right) \mathrm{Res}_z \sum_{j=1}^{n} \phi^{+(j)}(z)\tau_W(x,t) \otimes \phi^{-(j)}(z)\tau_W(x,t)\\
=&\mathrm{Res}_z \sum_{i,j=1}^{n} f_i(w)\phi^{\lambda(i)}(w)\phi^{+(j)}(z)\tau_W(x,t) \otimes \phi^{-(j)}(z)\tau_W(x,t)\\
=&\mathrm{Res}_z \sum_{i,j=1}^{n} f_i(w)\left(\delta_{\lambda,-}\delta_{ij}\delta(w-z) - \phi^{+(j)}(z)\phi^{\lambda(i)}(w)\right)\tau_W(x,t) \otimes \phi^{-(j)}(z)\tau_W(x,t).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{Res}_z \sum_{j=1}^{n} \phi^{+(j)}(z)\left(\sum_{i=1}^{n} f_i(w)\phi^{\lambda(i)}(w)\tau_W(x,t)\right) \otimes \phi^{-(j)}(z)\tau_W(x,t) =&\\
\delta_{\lambda,-}\tau_W(x,t) \otimes \sum_{i=1}^{n} f_i(w)\phi^{-(i)}(w)\tau_W(x,t)&
\end{aligned} \tag{79}$$

Next, let

$$1 \otimes \sum_{\ell=1}^{n} f_\ell(y)\phi^{\lambda(\ell)}(y),$$

act on (79), the action on left-hand side gives

$$\begin{aligned}
&\left(1 \otimes \sum_{\ell=1}^{n} f_\ell(y)\phi^{\lambda(\ell)}(y)\right) \mathrm{Res}_z \sum_{j=1}^{n} \phi^{+(j)}(z)\left(\sum_{i=1}^{n} f_i(w)\phi^{\lambda(i)}(w)\tau_W(x,t)\right) \otimes \phi^{-(j)}(z)\tau_W(x,t) =\\
&\mathrm{Res}_z \sum_{j,\ell=1}^{n} f_\ell(y)\phi^{+(j)}(z)\left(\sum_{i=1}^{n} f_i(w)\phi^{\lambda(i)}(w)\tau_W(x,t)\right) \otimes \left(\delta_{\lambda,+}\delta_{j,\ell}\delta(y-z) - \phi^{-(j)}(z)\phi^{\lambda(\ell)}(y)\right)\tau_W(x,t).
\end{aligned} \tag{80}$$

Its action on the right-hand side gives

$$\delta_{\lambda,-}\tau_W(x,t)\otimes\sum_{i,\ell=1}^{n} f_\ell(y)\phi^{-(\ell)}(y)f_i(w)\phi^{-(i)}(w)\tau_W(x,t) \tag{81}$$

Combining (80) and (81) we thus get

$$\begin{aligned}\mathrm{Res}_z\sum_{j=1}^{n}\phi^{+(j)}(z)\left(\sum_{i=1}^{n} f_i(w)\phi^{\lambda(i)}(w)\tau_W(x,t)\right)\otimes\phi^{-(j)}(z)\left(\sum_{\ell=1}^{n} f_\ell(y)\phi^{\lambda(\ell)}(y)\tau_W(x,t)\right) =\\ \delta_{\lambda,+}\sum_{i,\ell=1}^{n} f_\ell(y)\phi^{+(\ell)}(y)f_i(w)\phi^{+(i)}(w)\tau_W(x,t)\otimes\tau_W(x,t)\\ -\delta_{\lambda,-}\tau_W(x,t)\otimes\sum_{i,\ell=1}^{n} f_\ell(y)\phi^{-(\ell)}(y)f_i(w)\phi^{-(i)}(w)\tau_W(x,t).\end{aligned} \tag{82}$$

Define

$$\phi^{\pm}(f) := \mathrm{Res}_z\sum_{i=1}^{n} f_i(z)\phi^{\pm(i)}(z), \tag{83}$$

then

$$\phi^{\pm}(f) = \sum_{i=1}^{n}\sum_{k=-\infty}^{N_i} f_{ik}\phi^{\pm(i)}_{k+\frac{1}{2}}$$

for $f(z)$ given by (78). Now, take residues at w and y one obtains:

$$\mathrm{Res}_z\sum_{j=1}^{n}\phi^{+(j)}(z)\left(\phi^{\lambda}(f)\tau_W(x,t)\right)\otimes\phi^{-(j)}(z)\tau_W(x,t) = \delta_{\lambda,-}\tau_W(x,t)\otimes\phi^{-}(f)\tau_W(x,t) \tag{84}$$

and

$$\begin{aligned}\mathrm{Res}_z\sum_{j=1}^{n}\phi^{+(j)}(z)\left(\phi^{\lambda}(f)\tau_W(x,t)\right)\otimes\phi^{-(j)}(z)\left(\phi^{\lambda}(f)\tau_W(x,t)\right) =\\ \delta_{\lambda,+}\phi^{+}(f)\phi^{+}(f)\tau_W(x,t)\otimes\tau_W(x,t) - \delta_{\lambda,-}\tau_W(x,t)\otimes\phi^{-}(f)\phi^{-}(f)\tau_W(x,t).\end{aligned} \tag{85}$$

Applying lemma 7.1, we see that

$$\phi^{\pm}(f)\phi^{\pm}(f) = \mathrm{Res}_y\mathrm{Res}_z\frac{1}{2}\sum_{i,j=1}^{n} f_i(y)f_j(z)\left(\phi^{\pm(i)}(y)\phi^{\pm(j)}(z)+\phi^{\pm(j)}(z)\phi^{\pm(i)}(y)\right) = 0,$$

and hence, the right-hand side of (85) is 0. Thus $\phi^{\pm}(f)\tau_W(x,t)$ also satisfies the KP bilinear equation (77). In a similar way as one obtains formula (84), one can also deduce:

$$\mathrm{Res}_z\sum_{j=1}^{n}\phi^{+(j)}(z)\tau_W(x,t)\otimes\left(\phi^{-(j)}(z)\phi^{\lambda}(f)\tau_W(x,t)\right) = \delta_{\lambda,+}\phi^{+}(f)\tau_W(x,t)\otimes\tau_W(x,t). \tag{86}$$

We resume all this in the following

Proposition 7.2. *Let $\tau_W(x,t) = \sum_{\delta\in k\epsilon_1+\Lambda}\tau_W(\delta;x,t)e^{\alpha}$ be a KP tau-function corresponding to $W \in Gr^{(k)}(H)$ and let $f(z) = (f_1(z), f_2(z), \ldots, f_n(z))$ and $\phi^{\pm}(f)$ be defined by (83). Then $\phi^{\pm}(f)\tau_W(x,t)$ is also a KP tau-function, moreover $\tau_W(x,t)$ and $\phi^{\pm}(f)\tau_W(x,t)$ satisfy the first-modified KP hierarchy, i.e., equation (84) or (86).*

Clearly our new tau-function is of the form

$$\phi^{\pm}(f)\tau_W(x,t) = \sum_{\beta\in(k\pm1)\epsilon_1+\Lambda}\tau(\beta;x,t)e^{\beta}.$$

Assume that $\beta \in (k\pm1)\epsilon_1+\Lambda$ is of the form $\beta = \alpha\pm\epsilon_a$ for some $1\le a\le n$ with $\alpha\in k\epsilon_1+\Lambda$, then

$$\begin{aligned}\tau(\alpha\pm\epsilon_a;x,t)e^{\alpha\pm\epsilon_a} =& \left(\text{coefficient of } e^{\beta} \text{ in } \phi^{\pm}(f)\tau_W(x,t)\right)e^{\beta}\\ =&\mathrm{Res}_z\sum_{i=1}^{n} f_i(z)\phi^{\pm(i)}(z)\tau_W(\alpha\pm\epsilon_a\mp\epsilon_i;x,t)e^{\alpha\pm\epsilon_a\mp\epsilon_i}\\ =&\mathrm{Res}_z\sum_{i=1}^{n} f_i(z)\varepsilon(\epsilon_i,\alpha+\epsilon_a)z^{(\epsilon_i|\pm\alpha+\epsilon_a-\epsilon_i)}\\ &\times\exp(\pm\sum_{k=1}^{\infty}t_k^{(i)}z^k)\exp(\mp xz\mp\sum_{k=1}^{\infty}\frac{\partial}{\partial t_{ki}}\frac{z^{-k}}{k})\tau_W(\alpha\pm\epsilon_a-\epsilon_i;x,t)e^{\alpha\pm\epsilon_a}\\ =&\mathrm{Res}_z\sum_{i=1}^{n}\psi_W^{\pm}(\alpha;x,t,z)_{ai}f_i(z)\tau_W(\alpha;x,t)e^{\alpha\pm\epsilon_a}\\ =&q_a^{\pm}(\alpha;x,t,f)\tau_W(\alpha;x,t)e^{\alpha\pm\epsilon_a},\end{aligned}$$

where

$$\begin{aligned}q^{\pm}(\alpha;x,t,f) =& \left(q_1^{\pm}(\alpha;x,t,f), q_2^{\pm}(\alpha;x,t,f),\ldots,q_n^{\pm}(\alpha;x,t,f)\right)^T\\ :=&\mathrm{Res}_z\psi_W^{\pm}(\alpha;x,t,z)f(z)^T.\end{aligned}\tag{87}$$

Clearly,

$$q^{\pm}(\alpha;x,t,f) = \left(\frac{\tau(\alpha\pm\epsilon_1;x,t)}{\tau_W(\alpha;x,t)}, \frac{\tau(\alpha\pm\epsilon_2;x,t)}{\tau_W(\alpha;x,t)},\ldots,\frac{\tau(\alpha\pm\epsilon_n;x,t)}{\tau_W(\alpha;x,t)}\right)^T.\tag{88}$$

Note that since $\psi_W^{\pm}(\alpha;x,t,z)$ satisfies the linear system (16), (17) and its adjoint system (42), our $q^{\pm}(\alpha;t,f)$ satisfies

$$\begin{aligned}\frac{\partial q^+(\alpha;x,t,f)}{\partial t_{ki}} &= (L(\alpha;x,t,\partial)^k U_i(\alpha;x,t,\partial))_+ q^+(\alpha;x,t,f),\\ \frac{\partial q^-(\alpha;x,t,f)}{\partial t_{ki}} &= -(L(\alpha;x,t,\partial)^{*k} U_i(\alpha;x,t,\partial)^*)_+ q^-(\alpha;x,t,f).\end{aligned}\tag{89}$$

For this reason we call $q^+(\alpha;x,t,f)$ a *KP eigenfunction* and $q^-(\alpha;x,t,f)$ an *adjoint KP eigenfunction*.

It is now straightforward to find an expression for the new wave functions. Assume that $\lambda =$ either $+$ or $-$, then

$$\begin{aligned}
&\psi^{\pm}(\alpha+\lambda\epsilon_a;x,t,z)_{ij} = \\
&\varepsilon(\epsilon_j,\alpha+\lambda\epsilon_a+\epsilon_i)z^{(\epsilon_j|\pm\alpha+\lambda\epsilon_a+\epsilon_i-\epsilon_j)}\frac{\exp(\mp\sum_{k=1}^{\infty}\frac{\partial}{\partial t_{kj}}\frac{z^{-k}}{k})\tau(\alpha+\lambda\epsilon_a\pm\epsilon_i-\epsilon_j;x,t)}{\tau(\alpha+\lambda\epsilon_a;x,t)}\exp(\pm xz\pm\sum_{k=1}^{\infty}t_{kj}z^k) \\
=&\varepsilon_{ja}z^{\pm\lambda\delta_{ja}}\varepsilon(\epsilon_j,\alpha+\epsilon_i)z^{(\epsilon_j|\pm\alpha+\epsilon_i-\epsilon_j)} \\
&\times\frac{\exp(\mp\sum_{k=1}^{\infty}\frac{\partial}{\partial t_{kj}}\frac{z^{-k}}{k})q_a^{\lambda}(\alpha\pm(\epsilon_i-\epsilon_j);x,t,f)\tau_W(\alpha\pm(\epsilon_i-\epsilon_j);x,t)}{q_a^{\lambda}(\alpha;x,t,f)\tau_W(\alpha;x,t)}\exp(\pm xz\pm\sum_{k=1}^{\infty}t_k^{(j)}z^k) \\
=&\varepsilon_{ja}z^{\pm\lambda\delta_{ja}}\frac{q_a^{\lambda}(\alpha\pm(\epsilon_i-\epsilon_j);x,t_{k\ell}\mp\delta_{j\ell}\frac{z^{-k}}{k},f)}{q_a^{\lambda}(\alpha;x,t,f)}\psi_W^{\pm}(\alpha;x,t,z)_{ij}
\end{aligned} \tag{90}$$

Next, we want to calculate the operators which act ond the old wave functions and produce the new wave functions. For this purpose we rewrite equation (84) in terms of wave functions. We now want to calculate the coefficient of $e^{\beta+\epsilon_i}\otimes e^{\alpha-\epsilon_j}$ on both sides of (84) for arbitrary $\alpha\in k\epsilon_1+\Lambda$ and arbitrary $\beta\in(k+\lambda 1)\epsilon_1+\Lambda$. For this purpose it will be convenient to define the functions $q^{\pm}(\beta;x,t,f)$ by formula (88) for $\beta\in(k+\lambda 1)\epsilon_1+\Lambda$ also. So define

$$\begin{aligned}
&q^{\pm}(\beta;x,t,f)=(q_1^{\pm}(\beta;x,t,f),q_2^{\pm}(\beta;x,t,f),\dots,q_n^{\pm}(\beta;x,t,f))^T, \quad \text{where} \\
&q_i^{\pm}(\beta;x,t,f)=\frac{1}{q_i^{\mp}(\beta\pm\epsilon_i;x,t,f)},
\end{aligned}$$

then from (84) one deduces that

$$\mathrm{Res}_z\sum_{k=1}^{n}\psi^{+}(\beta;x,t,z)_{ik}\psi_W^{-}(\alpha;y,s,z)_{jk}=\delta_{\lambda,-}q_i^{+}(\beta;x,t,f)q_j^{-}(\alpha;y,s,f).$$

This gives the matrix bilinear equation:

$$\mathrm{Res}_z\psi^{+}(\beta;x,t,z)\psi_W^{-}(\alpha;y,s,z)^T=\delta_{\lambda,-}q^{+}(\beta;x,t,f)q^{-}(\alpha;y,s,f)^T. \tag{91}$$

Equation (86) leads to:

$$\mathrm{Res}_z\psi_W^{+}(\alpha;x,t,z)\psi^{-}(\beta;y,s,z)^T=\delta_{\lambda,+}q^{+}(\alpha;x,t,f)q^{-}(\beta;y,s,f)^T. \tag{92}$$

Using (91) when $\lambda=+$ and (92) when $\lambda=-$ it is easy to deduce the following proposition.

Proposition 7.3. *Let $\lambda=+$ or $-$, then $V\in Gr^{(k+\lambda 1)}(H)$ corresponding to the tau function $\tau_V=\phi^{\lambda}(f)\tau_W(x,t)$ is related to $W\in Gr^{(k)}(H)$ as follows:*

$$V\subset W \quad \text{if } \lambda=+, \qquad W\subset V \quad \text{if } \lambda=-.$$

Both inclusions are codimension 1 inclusions.

Now using the fundamental Theorem 4.1, equation (91) leads to

$$\left(P^+(\beta;x,t,\partial)R(\beta-\alpha,\partial)P^-(\alpha;x,t,\partial)^*\right)_- = \delta_{\lambda,-}q^+(\beta;x,t,f)\partial^{-1}q^-(\alpha;x,t,f)^T. \tag{93}$$

Assume as before that $\beta=\alpha+\lambda\epsilon_a$, then

$$\left(P^+(\alpha+\lambda\epsilon_a;x,t,\partial)R(\lambda\epsilon_a,\partial)P^+(\alpha;x,t,\partial)^{-1}\right)_- = \delta_{\lambda,-}q^+(\alpha-\epsilon_a;x,t,f)\partial^{-1}q^-(\alpha;x,t,f)^T. \tag{94}$$

Equation (92) gives with $\beta=\alpha+\lambda\epsilon_a$:

$$\left(P^+(\alpha;x,t,\partial)R(-\lambda\epsilon_a,\partial)P^+(\alpha+\lambda\epsilon_a,t,\partial)^{-1}\right)_- = \delta_{\lambda,+}q^+(\alpha;x,t,f)\partial^{-1}q^-(\alpha+\epsilon_a;x,t,f)^T. \tag{95}$$

Note that

$$\begin{aligned} &q^+(\alpha-\epsilon_a;x,t,f)= \\ &\quad\left(q_1^-(\alpha+\epsilon_1-\epsilon_a;x,t,f)^{-1},q_2^-(\alpha+\epsilon_2-\epsilon_a;x,t,f)^{-1},\ldots,q_n^-(\alpha+\epsilon_n-\epsilon_a;x,t,f)^{-1}\right)^T, \end{aligned}$$

and hence in this case $q^+(\beta;x,t,f)$ (and similarly $q^-(\beta;x,t,f)$) can be calculated using the old wave functions. It is now clear that in order to calculate the operator which acts on $\Psi^\pm(\alpha;x,t,z)$ and gives $\Psi^\pm(\alpha+\lambda\epsilon_a;x,t,z)$, we only have to calculate the differential part of

$$\begin{gathered} P^+(\alpha+\lambda\epsilon_a;x,t,\partial)R(\lambda\epsilon_a,\partial)P^+(\alpha;x,t,\partial)^{-1} \quad \text{and} \\ P^+(\alpha;x,t,\partial)R(-\lambda\epsilon_a,\partial)P^+(\alpha+\lambda\epsilon_a;x,,t,\partial)^{-1}. \end{gathered}$$

Since

$$R(\pm\lambda\epsilon_a,\partial)=\sum_{i=1}^n \delta_{ia}\partial^{\pm\lambda\epsilon_{ia}}E_{ii},$$

one finds:

$$\begin{aligned} \left(P^+(\alpha-\epsilon_a;x,t,\partial)R(-\epsilon_a,\partial)P^+(\alpha;x,t,\partial)^{-1}\right)_+ &= \left(P^+(\alpha;x,t,\partial)R(-\epsilon_a,\partial)P^+(\alpha-\epsilon_a;x,t,\partial)^{-1}\right)_+ \\ &=\sum_{i\neq a}\epsilon_{ia}E_{ii} \end{aligned} \tag{96}$$

and

$$\begin{aligned} &\left(P^+(\alpha+\epsilon_a;x,t,\partial)R(\epsilon_a,\partial)P^+(\alpha;x,t,\partial)^{-1}\right)_+ \\ &\qquad=\partial E_{aa}+\sum_{i\neq a}\epsilon_{ia}E_{ii}+\sum_{j=1}^n W_{ja}^+(\alpha+\epsilon_a;x,t)E_{ja}-W_{aj}^+(\alpha;x,t)E_{aj}, \\ &\left(P^+(\alpha;x,t,\partial)R(\epsilon_a,\partial)P^+(\alpha-\epsilon_a;x,t,\partial)^{-1}\right)_+ \\ &\qquad=\partial E_{aa}+\sum_{i\neq a}\epsilon_{ia}E_{ii}+\sum_{j=1}^n W_{ja}^+(\alpha;x,t)E_{ja}-W_{aj}^+(\alpha-\epsilon_a;x,t)E_{aj}, \end{aligned} \tag{97}$$

where $W^+(\delta;x,t)$ is defined by (70). It is straightforward to check, using (88) and (74) that

$$\begin{aligned} W_{ja}^+(\alpha;x,t) &=\epsilon_{aj}q_j^+(\alpha-\epsilon_a;x,t,f)q_a^-(\alpha;x,t,f), \quad j\neq a \\ W_{aj}^+(\alpha;x,t) &=\epsilon_{ja}q_a^+(\alpha;x,t,f)q_j^-(\alpha+\epsilon_a;x,t,f), \quad j\neq a \\ W_{aa}^+(\alpha;x,t)-W_{aa}^+(\alpha\pm\epsilon_a;x,t) &=\frac{\partial\log q_a^\pm(\alpha;x,t,f)}{\partial t_{1a}}. \end{aligned} \tag{98}$$

Using (98) and sometimes taking the adjoint we find

Proposition 7.4.

$$\psi^{\pm}(\alpha \mp \epsilon_a; x, t, z) = \left(\sum_{i \neq a} \epsilon_{ia} E_{ii} \pm q^{\pm}(\alpha \mp \epsilon_a; x, t, f) \partial^{-1} q^{\mp}(\alpha; x, t, f)^T \right) \psi_W^{\pm}(\alpha; t, z)$$

$$\psi^{\pm}(\alpha \pm \epsilon_a; x, t, z) = \Bigg(\pm \left(\partial - \frac{\partial \log q_a^{\pm}(\alpha; x, t, f)}{\partial t_{1a}} \right) E_{aa} + \sum_{i \neq a} \epsilon_{ia} E_{ii}$$

$$+ \sum_{j \neq a} \epsilon_{aj} \left(q_a^{\pm}(\alpha; x, t, f) q_j^{\mp}(\alpha \pm \epsilon_a; x, t, f) E_{aj} + q_j^{\pm}(\alpha; x, t, f) q_a^{\mp}(\alpha \pm \epsilon_a; x, t, f) E_{ja} \right) \Bigg) \psi_W^{\pm}(\alpha; t, z)$$

Using (69) one easily sees the following consequences

Corollary 7.1.

$$\psi^{\pm}(\beta; x, t, z) = \pm \sum_{a=1}^{n} \left(\sum_{j=1}^{n} q_a^{\pm}(\beta; x, t, f) \partial^{-1} q_j^{\mp}(\beta \pm \epsilon_a; x, t, f) E_{aj} \right) \psi_W^{\pm}(\beta \pm \epsilon_a; x, t, z),$$

$$\psi^{\pm}(\beta; x, t, z) = \pm \sum_{a=1}^{n} \left(\frac{\partial}{\partial t_{1a}} - \frac{\partial \log q_a^{\pm}(\beta \mp \epsilon_a; x, t, f)}{\partial t_{1a}} \right) E_{aa} \psi_W^{\pm}(\beta \mp \epsilon_a; x, t, z).$$

The second equation of this corollary and proposition 7.3 gives the following geometric result:

Theorem 7.1. *Let* $\tau_W(x,t) = \sum_{\alpha \in k\epsilon_1 + \Lambda} \tau_W(\alpha; x, t) e^{\alpha}$ *be a KP tau-function corresponding to* $W \in Gr^{(k)}(H)$*, then*

$$\tau_V(x,t) := \phi^{\lambda}(f) \tau_W(x,t) = \sum_{\beta \in (k+\lambda 1)\epsilon_1 + \Lambda} \tau_V(\beta; x, t) e^{\beta},$$

is a KP tau-function corresponding to $V \in Gr^{(k+\lambda 1)}(H)$*. For all* $\beta \in (k + \lambda 1)\epsilon_1 + \Lambda$ *the (adjoint) wave functions satisfy*

$$Res_z \Psi_V^{\lambda}(\beta; x, t, z) f(z)^T = 0,$$

which means that

$$V = \{w \in W | Res_z w(z) f(z)^T = 0\} \qquad \text{if } \lambda = +,$$
$$V^{\perp} = \{w \in W^{\perp} | Res_z w(z) f(z)^T = 0\} \qquad \text{if } \lambda = -.$$

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